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SUBHARMONIC FUNCTIONS ON GRAPHS

BY

MARCO RIGOLI, MAURA SALVATORI AND MARCO VIGNATI

Dipartimento di Matematica, Università di Milano
via Saldini 50, 20133 Milano, Italy
e-mail: rigoli@vmimat.mat.unimi.it
mauras@vmimat.mat.unimi.it vignati@vmimat.mat.unimi.it

ABSTRACT

We study the behaviour of subharmonic functions on a graph. We assume bounds on the growth of balls and functions in order to obtain Liouville type theorems.

0. Introduction

The aim of this paper is to analyse the qualitative behaviour of subharmonic functions on graphs. To fix notations, G shall denote an infinite, connected graph, with uniformly bounded vertex degree.

It turns out that, in a number of basic questions, the asymptotic growth of the cardinality of balls plays a preminent rôle among the most simple structural properties of G. Somehow, to parallel the continuous case, G compares with a complete manifold whose geometry is controlled in terms of volume, avoiding curvature assumptions.

The basic question to which we presently address concerns uniqueness, up to constants, of solutions of the differential inequality

$$\Delta u \geq 0 \qquad \text{on } G. \tag{0.1}$$

Otherwise said, we look for a type of result reminiscent of the classical Liouville theorem.

Motivated by the works of Karp, [K], and Dodziuk and Karp, [D–K], we begin with analysing the ℓ^p behaviour of solutions of (0.1). We prove

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THEOREM A: *Let u be a non-negative subharmonic function on G. Then, either u is constant or, for any $2 \le p < +\infty$,*

$$\lim_{R\to+\infty} \frac{1}{R^2} \sum_{B_R(q)} u^p(x) = +\infty.$$

Here, $B_R(q)$ is the ball of radius R centered at the vertex q. Note that the choice of q plays no rôle.

As a matter of fact, the conclusion of Theorem A is bothered with the unpleasant restriction $p \ge 2$. Indeed, one would expect, as a natural range for p, the interval $(1, +\infty)$. It seems a challenging problem to determine whether this assumption is a technical weakness of the argument presented below, or it pertains to the nature of the discrete Laplacian. However, we obtain a partial, significative answer to this question in Theorem C of section 3. A meaningful consequence of this latter result is contained in

COROLLARY D: *Let $0 < \delta < 1$ and assume that the graph G satisfies*

$$\liminf_{R\to+\infty} \frac{|B_R(q)|}{R^{2-\delta}} = 0 \tag{0.2}$$

for some vertex q. Then any subharmonic function u such that

$$u(x) \le A\rho^k(x,q) + B$$

for some constants $A, B > 0$, $0 \le k < \delta$, is constant.

Here, $|\cdot|$ denotes cardinality and ρ the distance in the graph.

The aim of section 4 is to show that (0.2) can be relaxed as to allow $\delta = 0$, in Corollary D, provided u is at most of logarithmic growth. Indeed, we prove

THEOREM E: *Suppose that, for some vertex q,*

$$\lim_{R\to+\infty} \frac{|B_R(q)|}{R^2} = 0. \tag{0.3}$$

Let u be a subharmonic function such that

$$u(x) \le A \log \rho(x,q) + B \tag{0.4}$$

for some constants $A, B > 0$. Then u is constant.

To the best of our knowledge even the continuous version of Theorem E is new. Its proof, on a complete Riemannian manifold, can be modeled after the arguments presented below.

In literature, Liouville type theorems can be divided into two major classes: those involving bounds on the functions, and those involving bounds on the energy. In particular, into the former class fall a number of results concerning recurrence (equivalently parabolicity) of graphs. We refer the reader to the survey article –and references therein– of Woess [W]. Liouville theorems of the second type have been proved, in the discrete case, by a number of authors. Notably we mention the papers of Soardi [S1], Cartwright and Woess [C–W] and Benjamini and Schramm [B–S]. Again we refer to Woess [W] and to the recent book of Soardi [S2].

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1. Preliminaries and notations

Let G be an infinite graph with set of vertices X, and set of edges E. Thus, $|X| = \infty$ and E is a symmetric, non-reflexive relation, $E \subset X \times X$. Throughout the paper, the notation $|B|$ stands for cardinality of the set B. For $(x, y) \in E$ we shall write $x \sim y$ and say that x and y are **neighbours**. We shall not be concerned with the orientation of edges.

Given a vertex x we let

$$m(x) = |\{y \in X: y \sim x\}|\,. \tag{1.1}$$

We assume that the graph has uniformly bounded vertex degree, that is, there exists $M > 0$ such that

$$m(x) \leq M \qquad \text{for each } x \in X. \tag{1.2}$$

A **path** σ, joining vertices x and y, is a (finite) sequence $x_0 = x, x_1, \ldots, x_n = y$ of vertices such that $x_i \sim x_{i+1}$. Obviously, $l(\sigma) = n$ is the length of σ.

G is a **connected** graph if every pair of vertices can be joined by a path. In this case we introduce a **distance** ρ: $X \times X \to \mathbb{N}$ by setting $\rho(x, x) = 0$, and for $x \neq y$,

$$\rho(x, y) = \inf l(\sigma) \tag{1.3}$$

where the infimum is taken over all paths joining x and y.

For $q \in X$, $R \in \mathbb{N}$, we denote

$$B_R(q) = \{x \in X: \rho(x,q) \le R\}, \quad S_R(q) = \{x \in X: \rho(x,q) = R\}. \tag{1.4}$$

A **vector field** F is a map $F: E \to \mathbb{R}$ with the property

$$F(x,y) = -F(y,x) \qquad \text{for every } x \sim y. \tag{1.5}$$

Given vector fields F and G, their **inner product** is the function $F \cdot G: X \to \mathbb{R}$ defined by

$$(F \cdot G)(x) = \sum_{y \sim x} F(x,y) G(x,y). \tag{1.6}$$

Thus, $\|F\|$, the **norm** of F, is

$$\|F\|(x) = (F \cdot F)^{1/2}(x) = \left(\sum_{y \sim x} [F(x,y)]^2 \right)^{1/2}. \tag{1.7}$$

Let $f: X \to \mathbb{R}$ be a function and F a vector field. The **scalar product** $(f \star F)$ is the vector field

$$(f \star F)(x,y) = \frac{1}{2}[f(x) + f(y)] F(x,y). \tag{1.8}$$

The **divergence** and **gradient** operators allow us to produce functions from vector fields and vice versa according to the following rules:

$$(\operatorname{div} F)(x) = \frac{1}{m(x)} \sum_{y \sim x} F(x,y), \tag{1.9}$$

$$(df)(x,y) = f(y) - f(x). \tag{1.10}$$

A simple computation yields

$$\operatorname{div}(f \star F)(x) = f(x) \operatorname{div} F(x) + \frac{1}{2m(x)} (df \cdot F)(x). \tag{1.11}$$

Given a function $f: X \to \mathbb{R}$ define its **Laplacian** by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \sim x} f(y) - f(x). \tag{1.12}$$

It is an easy matter to check that

$$\Delta f(x) = (\operatorname{div} df)(x). \tag{1.13}$$

We say that the function f is **harmonic** (resp. **subharmonic**) if $\Delta f = 0$ ($\Delta f \geq 0$).

Observe that, in case $F = dg$, for some $g\colon X \to \mathbb{R}$, (1.11) becomes

$$\operatorname{div}(f \star dg)(x) = f(x)\Delta g(x) + \frac{1}{2m(x)}(df \cdot dg)(x). \tag{1.14}$$

It is immediate to verify the validity of the following discrete analogue of **Green's formula.**

PROPOSITION 1.1 ([D–K]): *If N is a finite subset of X and $u, v\colon X \to \mathbb{R}$, then*

$$\begin{aligned} &\frac{1}{2}\sum_{\substack{x,y\in N\\ x\sim y}} [u(y)-u(x)][v(y)-v(x)] \\ &= -\sum_{x\in N} m(x)v(x)\Delta u(x) + \sum_{x\in N}\sum_{\substack{y\in\partial N\\ y\sim x}} v(x)[u(y)-u(x)] \end{aligned} \tag{1.15}$$

where $\partial N = \overline{N} \setminus N$ and $\overline{N} = \{y \in X\colon \exists x \in N, x \sim y\}$.

In the sequel we shall often use the notion of cut-off function. For the sake of clarity we introduce the following

Definition: Let $R > r + 1$ be integers, $q \in X$. A **cut-off** function φ associated to the region $C_{R,r}(q) = \{x \in X\colon r \leq \rho(x,q) \leq R\}$ is a function $\varphi\colon X \to [0,1]$ satisfying the following requirements:

(1) $\varphi(x) = 1$ when $\rho(x,q) \leq r$ and $\varphi(x) = 0$ when $\rho(x,q) \geq R$;

(2) there exists a constant $c > 0$ such that

$$\|d\varphi\|(x) \leq \frac{c}{R-r} \qquad \text{for every vertex } x \in X. \tag{1.16}$$

Observe that $C_{R,0}(q) = B_R(q)$.

Since G has uniformly bounded vertex degree, it is possible to guarantee the existence of such functions. To give an example,

$$\varphi(x) = \begin{cases} 1 & \text{when } \rho(x,q) \leq r \\ \frac{R-\rho(x,q)}{R-r} & r \leq \rho(x,q) \leq R \\ 0 & \text{when } \rho(x,q) \geq R \end{cases} \tag{1.17}$$

is a radial, with respect to q, piecewise linear, cut-off function for $C_{R,r}(q)$.

We note that this choice of φ satisfies

$$\varphi(x) \leq 2\varphi(y) \qquad \text{for every } x \sim y \in B_{R-1}(q). \tag{1.18}$$

Green's formula yields the following version of the **divergence theorem:**

PROPOSITION 1.2: *Let u: $X \to \mathbb{R}$ and let φ be a cut-off function for the region $C_{R,r}(q)$. Then*

$$\sum_{B_R(q)} m(x)\,\mathrm{div}(\varphi \star du)(x) = 0. \tag{1.19}$$

Proof: From Green's formula and the properties of φ we have

$$\begin{aligned}\sum_{B_R(q)} m(x)\,\mathrm{div}(\varphi \star du)(x) &= \sum_{B_R(q)} m(x)\varphi(x)\Delta u(x) + \frac{1}{2}\sum_{B_R(q)} (d\varphi{\cdot}du)(x)\\ &= \sum_{B_R(q)} m(x)\varphi(x)\Delta u(x) + \frac{1}{2}\sum_{x\in B_R(q)}\sum_{\substack{y\sim x\\ y\in B_R(q)}} du(x,y)d\varphi(x,y)\\ &\quad + \frac{1}{2}\sum_{\rho(x,q)=R}\sum_{\substack{y\sim x\\ \rho(y,q)=R+1}} du(x,y)[\varphi(y)-\varphi(x)]\\ &= \sum_{x\in B_R(q)}\sum_{\substack{y\sim x\\ y\in\partial B_R(q)}} \varphi(x)[u(y)-u(x)] = 0. \quad ∎\end{aligned}$$

Remark: The radius r plays no rôle.

Throughout the paper we shall assume that G is infinite, connected, with uniformly bounded vertex degree. Furthermore, c will always denote a positive absolute constant which may change from line to line, unless otherwise specified.

2. Non-existence of ℓ^p-subharmonic functions. The case $p \geq 2$

The main result of this section shows that no non-negative subharmonic function belongs to $\ell^p(X)$ for $p \geq 2$. (For a more precise statement, see Theorem A below.) As remarked in the Introduction, we are able to obtain conclusion (2.1) only for $p \geq 2$. We conjecture that this apparently technical assumption, rather than the most natural $1 < p < \infty$, is due to the intimate nature of the discrete Laplacian. More precisely, we believe it concerns the fact that, in the discrete case, the Laplacian involves only differences of the first order. A weaker, nevertheless interesting conclusion in case $1 < p < 2$, is contained in Theorem C of section 3. As an immediate consequence of Theorem A we obtain a sufficient condition to guarantee recurrence of a graph.

We should mention that Karp [K] proves a fairly general continuous version of Theorem A on a complete, non-compact, Riemannian manifold. It will become apparent to the reader that our argument is based on the same lines of reasoning. However, the discrete case reveals some extra difficulties which make the present proof a not at all obvious generalization of Karp's. (See also the special case, in the discrete setting, treated by Dodziuk and Karp [D–K].)

THEOREM A: *Let $u: X \to [0,+\infty)$ be a non-constant, subharmonic function. Then, for any fixed vertex q and $p \geq 2$,*

$$\lim_{R\to+\infty} \frac{1}{R^2} \sum_{B_R(q)} u^p(x) = +\infty. \tag{2.1}$$

The proof of Theorem A (as well as that of Theorem C of section 3) employs estimates of the following type:

LEMMA 2.1: *Let $u, g: X \to \mathbb{R}$ and φ be a cut-off function for the region $C_{R,r}(q)$. Then*

(i) $$\sum_{B_R(q)} \|(\varphi g) \star du\|^2(x) \leq \sum_{B_R(q)} \varphi^2(x) g^2(x) \|du\|^2(x);$$

(ii) $$\sum_{B_R(q)} \|g \star d\varphi\|^2(x) \leq \sum_{B_R(q)} g^2(x) \|d\varphi\|^2(x).$$

Proof: (i) We compute

$$\sum_{B_R(q)} 4\|(\varphi g) \star du\|^2(x) = \sum_{x\in B_R(q)} \sum_{y\sim x} [(\varphi g)(x) + (\varphi g)(y)]^2 [du(x,y)]^2$$
$$\leq 2 \sum_{x\in B_R(q)} \sum_{y\sim x} [(\varphi g)^2(x) + (\varphi g)^2(y)][du(x,y)]^2$$
$$= 2 \sum_{B_R(q)} (\varphi g)^2(x) \|du\|^2(x) + 2 \sum_{x\in B_R(q)} \sum_{y\sim x} (\varphi g)^2(y)[du(x,y)]^2.$$

Next, we observe that, since φ is null outside $B_R(q)$, the sum in the last term of the inequality can be performed on the symmetric set

$$\{x, y \in B_R(q): x \sim y\}.$$

Thus we can exchange x and y, and then dominate with

$$\sum_{y\in B_R(q)}\sum_{x\sim y}(\varphi g)^2(y)[du(x,y)]^2$$

obtaining (i).

(ii) Since φ is a cut-off function for $C_{R,r}(q)$ we have

$$\|d\varphi\|(y)=0 \qquad \text{on } S_{R+1}(q). \tag{2.2}$$

We compute

$$\begin{aligned}
\sum_{B_R(q)} 4\|g\star d\varphi\|^2(x) &\le 2\sum_{x\in B_R(q)}\sum_{y\sim x}[g^2(x)+g^2(y)][d\varphi(x,y)]^2\\
&= 2\sum_{B_R(q)} g^2(x)\|d\varphi\|^2(x)+2\sum_{x\in B_R(q)}\sum_{y\sim x} g^2(y)[d\varphi(x,y)]^2\\
&\le 2\sum_{x\in B_R(q)} g^2(x)\|d\varphi\|^2(x)+2\sum_{y\in B_{R+1}(q)} g^2(y)\sum_{y\sim x}[d\varphi(x,y)]^2\\
&\le 2\sum_{x\in B_R(q)} g^2(x)\|d\varphi\|^2(x)+2\sum_{y\in B_{R+1}(q)} g^2(y)\|d\varphi\|^2(y)\\
&= 4\sum_{B_R(q)} g^2(x)\|d\varphi\|^2(x)
\end{aligned}$$

where in the last step we have used (2.2). ∎

LEMMA 2.2: *Let $u\colon X\to[0,+\infty)$ be a subharmonic function and $p\ge 2$. Then*

$$m(x)\Delta u^p(x)\ge u^{p-2}(x)\|du\|^2(x) \tag{2.3}$$

for each $x\in X$.

Proof: We fix $x\in X$. If $u(x)=0$ and $p=2$ formula (2.3) is trivial, while for $p>2$ the condition $u\ge 0$ implies $\Delta u^p\ge 0$ so that (2.3) holds. Otherwise, (2.3) amounts to showing the validity of the inequality

$$\sum_{y\sim x}\left\{\left[\frac{u(y)}{u(x)}\right]^p-1\right\}\ge\sum_{y\sim x}\left\{\frac{u(y)}{u(x)}-1\right\}^2 \tag{2.4}$$

under the assumption

$$\frac{1}{m(x)}\sum_{y\sim x}u(y)\ge u(x). \tag{2.5}$$

This is indeed the case since

$$\frac{\sum_{j=1}^{m}(z_j^p - 1)}{\sum_{j=1}^{m}(z_j - 1)^2} \geq 1, \qquad m = m(x), \qquad p \geq 2$$

on

$$\{\sum_{j=1}^{m} z_j \geq m, \quad z_j \geq 0, \quad j = 1, \ldots, m\}.$$

This follows from the inequality $z^p - 1 \geq p(z-1) + \frac{p}{2}(z-1)^2$ valid for $z \geq 0$ and $p \geq 2$. ∎

Remark: Note that (2.3) is false when $1 < p < 2$.

Proof (of Theorem A): We fix a vertex q in G and $p \geq 2$. Next, we observe that the result follows at once from the validity of the following claim: let $R \geq r + 2$ and φ be a linear, radial, cut-off function for the region $C_{R,r}(q)$ as in (1.17), then

$$\left\{ \sum_{B_R(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x) \right\}^2 \leq \frac{c}{(R-r)^2} \left\{ \sum_{B_R(q)} u^p(x) \right\} \left\{ \sum_{C_{R,r-1}(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x) + \frac{1}{(R-r)^2} \sum_{x \in S_R} \sum_{\substack{y \sim x \\ y \in S_{R-1}}} u^{p-2}(x)[u(y) - u(x)]^2 \right\}. \tag{2.6}$$

Indeed, by contradiction assume that

$$\liminf_{R \to +\infty} \frac{1}{R^2} \sum_{B_R(q)} u^p < +\infty.$$

This means that we can find a sequence $\{R_k\}$ such that

$$R_k \geq 2R_{k-1} \tag{2.7}$$

and $B \in (0, +\infty)$ so that

$$\frac{1}{R_k^2} \sum_{B_{R_k}(q)} u^p \leq B \qquad \text{for each } k. \tag{2.8}$$

For each fixed k we define

$$A_k = \frac{1}{R_k^2} \sum_{B_{R_k}(q)} u^p, \tag{2.9}$$

$$r_k = R_{k-1} + 2, \tag{2.10}$$

$$\beta_k = \frac{1}{(R_k - r_k)^2} \sum_{x \in S_{R_k}} \sum_{\substack{y \sim x \\ y \in S_{R_k - 1}}} u^{p-2}(x)[u(x) - u(y)]^2, \tag{2.11}$$

and, for the radial, linear cut-off function φ_k associated to the region $C_{R_k, r_k}(q)$ as in (1.17), we set

$$Q_k = \sum_{B_{R_k}(q)} \varphi_k^2(x) u^{p-2}(x) \|du\|^2(x). \tag{2.12}$$

Then (2.6) can be rewritten as

$$Q_k^2 \le c \frac{R_k^2}{(R_k - r_k)^2} A_k \left\{ \sum_{C_{R_k, r_k - 1}(q)} \varphi_k^2(x) u^{p-2}(x) \|du\|^2(x) + \beta_k \right\}$$

so that, with the aid of (2.7),

$$Q_k^2 \le c A_k \left\{ \sum_{C_{R_k, r_k - 1}(q)} \varphi_k^2(x) u^{p-2}(x) \|du\|^2(x) + \beta_k \right\}. \tag{2.13}$$

On the other hand the choice of r_k in (2.10), together with $\varphi_{k-1} \le \varphi_k$, gives

$$\sum_{C_{R_k, r_k - 1}(q)} \varphi_k^2(x) u^{p-2}(x) \|du\|^2(x) \le Q_k - Q_{k-1}. \tag{2.14}$$

Putting together (2.13) and (2.14) we finally obtain

$$Q_k^2 \le cB(Q_k - Q_{k-1} + \beta_k). \tag{2.15}$$

Since u is non-constant, certainly $Q_k > 0$ for $k \ge k_0$, and $Q_k \uparrow Q \in (0, +\infty]$. We want to show that (2.15) implies $Q = 0$, thus obtaining a contradiction. Towards this end we observe that, for any $R \in \mathbf{N}$,

$$\sum_{x \in S_R} \sum_{\substack{y \sim x \\ y \in S_{R-1}}} u^{p-2}(x)[u(x) - u(y)]^2 \le M \sum_{S_R \cup S_{R-1}} u^p(x). \tag{2.16}$$

To establish (2.16) we make use of $u \geq 0$, and of the elementary inequality

$$a^{p-2}(a-b)^2 \leq (a^p + b^p)$$

valid for $a, b \geq 0$ and $p \geq 2$.

Thus (2.16), (2.7) and (2.8) imply

$$\beta_k \leq cA_k \leq cB. \tag{2.17}$$

Substituting (2.17) in (2.15) and recalling that $Q_k > 0$ for k large, we obtain

$$Q_k \leq cB\left(1 + \frac{B}{Q_k}\right). \tag{2.18}$$

This proves that Q is finite; in particular

$$Q_k - Q_{k-1} \to 0 \qquad \text{as } k \to +\infty. \tag{2.19}$$

Moreover,

$$\beta_k \leq c\frac{Q}{(R_k - r_k)^2} \leq C\frac{Q}{R_{k-1}^2} \to 0 \quad \text{as } k \to +\infty; \tag{2.20}$$

now, (2.15), (2.19) and (2.20) imply

$$Q = 0$$

which contradicts our assumptions.

It remains to prove (2.6). We divide the argument in two steps.

STEP 1: Let R, r and φ be as in (2.6). From the divergence theorem

$$\sum_{B_R(q)} m(x)\,\mathrm{div}(\varphi^2 \star du^p)(x) = 0.$$

Hence (1.14) implies

$$\sum_{B_R(q)} m(x)\varphi^2(x)\Delta u^p(x) = -\frac{1}{2}\sum_{B_R(q)} (d\varphi^2 \cdot du^p)(x). \tag{2.21}$$

Applying (2.3) of Lemma 2.2 we then have

$$\sum_{B_R(q)} \varphi^2(x)u^{p-2}(x)\left\|du\right\|^2(x) \leq -c\sum_{B_R(q)} (d\varphi^2 \cdot du^p)(x). \tag{2.22}$$

We square (2.22), apply the Cauchy–Schwartz inequality, Lemma 2.1 (ii), and the properties of φ to obtain the following chain of inequalities:

$$
\begin{aligned}
&\Big\{\sum_{B_R(q)} \varphi^2(x)u^{p-2}(x)\,\|du\|^2(x)\Big\}^2 \le c\Big\{\sum_{B_R(q)} (d\varphi^2\cdot du^p)(x)\Big\}^2\\
&= c\Big\{\sum_{C_{R,r}(q)} (2\varphi\star d\varphi)\cdot(du^p)(x)\Big\}^2\\
&\le c\Big\{\sum_{x\in C_{R,r}(q)}\sum_{y\sim x}[\varphi(x)+\varphi(y)][u^{\frac{p-2}{2}}(x)+u^{\frac{p-2}{2}}(y)]\cdot\\
&\qquad\cdot|u(x)-u(y)|\,[u^{p/2}(x)+u^{p/2}(y)]\,|\varphi(x)-\varphi(y)|\Big\}^2\\
&= c\Big\{\sum_{x\in C_{R,r}(q)}\sum_{y\sim x}\Big|\varphi\star(u^{\frac{p-2}{2}}\star du)(x,y)\Big|\,\Big|(u^{p/2}\star d\varphi)(x,y)\Big|\Big\}^2\\
&\le c\Big\{\sum_{x\in C_{R,r}(q)}\Big\|\varphi\star(u^{\frac{p-2}{2}}\star du)\Big\|(x)\,\Big\|(u^{p/2}\star d\varphi)\Big\|(x)\Big\}^2\\
&\le c\Big\{\sum_{B_R(q)}\Big\|(u^{p/2}\star d\varphi)\Big\|^2(x)\Big\}\Big\{\sum_{x\in C_{R,r}(q)}\Big\|\varphi\star(u^{\frac{p-2}{2}}\star du)\Big\|^2(x)\Big\}\\
&\le \frac{c}{(R-r)^2}\Big\{\sum_{B_R(q)} u^p(x)\Big\}\Big\{\sum_{x\in C_{R,r}(q)}\Big\|\varphi\star(u^{\frac{p-2}{2}}\star du)\Big\|^2(x)\Big\}.
\end{aligned}
\tag{2.23}
$$

Step 2: We estimate the last term on the RHS of (2.23) from above. We compute

$$
\begin{aligned}
&\sum_{x\in C_{R,r}(q)}\Big\|\varphi\star(u^{\frac{p-2}{2}}\star du)\Big\|^2(x)\\
&= c\sum_{x\in C_{R,r}(q)}\sum_{y\sim x}[\varphi(x)+\varphi(y)]^2\Big[u^{\frac{p-2}{2}}(x)+u^{\frac{p-2}{2}}(y)\Big]^2[u(x)-u(y)]^2\\
&\le c\sum_{x\in C_{R,r}(q)}\sum_{y\sim x}[\varphi^2(x)+\varphi^2(y)]\,[u^{p-2}(x)+u^{p-2}(y)]\,[u(x)-u(y)]^2\\
&\le c\{\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}\}
\end{aligned}
\tag{2.24}
$$

with

$$\mathcal{A} = \sum_{x \in C_{R,r}(q)} \varphi^2(x) u^{p-2}(x) \left\{ \sum_{y \sim x} [u(x) - u(y)]^2 \right\}, \tag{2.25}$$

$$\mathcal{B} = \sum_{x \in C_{R,r}(q)} \sum_{y \sim x} \varphi^2(y) u^{p-2}(y) [u(x) - u(y)]^2, \tag{2.26}$$

$$\mathcal{C} = \sum_{x \in C_{R,r}(q)} \varphi^2(x) \left\{ \sum_{y \sim x} u^{p-2}(y) [u(x) - u(y)]^2 \right\}, \tag{2.27}$$

$$\mathcal{D} = \sum_{x \in C_{R,r}(q)} u^{p-2}(x) \left\{ \sum_{y \sim x} \varphi^2(y) [u(x) - u(y)]^2 \right\}. \tag{2.28}$$

To take care of $\mathcal{A}$ observe that

$$\mathcal{A} \le \sum_{C_{R,r-1}(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x) \tag{2.29}$$

and similarly

$$\mathcal{B} \le \sum_{C_{R,r-1}(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x). \tag{2.30}$$

As for $\mathcal{C}$ we observe that choice (1.17) of $\varphi(x)$ implies (1.18) to the effect that we can dominate $\varphi^2(x)$ with $4\varphi^2(y)$ whenever $x, y \in B_{R-1}(q)$ and $y \sim x$. Hence we obtain

$$\begin{aligned} \mathcal{C} \le c & \sum_{C_{R,r-1}(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x) \\ & + \sum_{x \in S_{R-1}(q)} \sum_{\substack{y \sim x \\ y \in S_R(q)}} \varphi^2(x) u^{p-2}(y) [u(x) - u(y)]^2. \end{aligned} \tag{2.31}$$

For $x \in S_{R-1}(q)$,

$$\varphi^2(x) \le \frac{c}{(R-r)^2}$$

and so

$$\begin{aligned} \mathcal{C} \le c & \sum_{C_{R,r-1}(q)} \varphi^2(x) u^{p-2}(x) \left\| du \right\|^2 (x) \\ & + \frac{c}{(R-r)^2} \sum_{x \in S_{R-1}(q)} \sum_{\substack{y \sim x \\ y \in S_R(q)}} u^{p-2}(y) [u(x) - u(y)]^2. \end{aligned} \tag{2.32}$$

On the other hand, the same procedure yields

$$\mathcal{D} \leq c \sum_{C_{R,r-1}(q)} \varphi^2(x)u^{p-2}(x) \left\|du\right\|^2(x) + \frac{c}{(R-r)^2} \sum_{x\in S_R(q)} \sum_{\substack{y\sim x \\ y\in S_{R-1}(q)}} u^{p-2}(x)[u(x)-u(y)]^2. \tag{2.33}$$

From (2.24), (2.29), (2.30), (2.32) and (2.33) we finally have

$$\begin{aligned} \sum_{x\in C_{R,r}(q)} & \left\|\varphi \star (u^{\frac{p-2}{2}} \star du)\right\|^2(x) \\ & \leq c \sum_{C_{R,r-1}(q)} \varphi^2(x)u^{p-2}(x) \left\|du\right\|^2(x) \\ & \quad + \frac{c}{(R-r)^2} \sum_{x\in S_R(q)} \sum_{\substack{y\sim x \\ y\in S_{R-1}(q)}} u^{p-2}(x)[u(x)-u(y)]^2. \end{aligned} \tag{2.34}$$

Putting together Steps 1 and 2 we derive (2.6). ∎

A simple application of Jensen's inequality and Theorem A gives

COROLLARY B: *Suppose that for some (hence any) $q \in X$ the graph G satisfies*

$$\liminf_{R\to+\infty} \frac{|B_R(q)|}{R^2} < +\infty. \tag{2.35}$$

Then the graph is recurrent.

Remark: We observe that (2.35) implies that $\sum_{R=1}^{+\infty} \frac{1}{|S_R(q)|} = +\infty$. Thus the conclusion of Corollary B also follows directly from the Nash–Williams criterion [N-W].

Proof: We need to show that any non-positive subharmonic function $u\colon X \to \mathbb{R}$ is constant. Towards this aim we define

$$v = \exp u.$$

Obviously, $0 < v \leq 1$ and, by Jensen's inequality,

$$\Delta v \geq 0.$$

Let $\{R_j\} \uparrow +\infty$ be a sequence satisfying

$$\frac{|B_{R_j}(q)|}{R_j^2} \to B < +\infty \qquad \text{as } j \to +\infty.$$

For any fixed $p \geq 2$ we then have

$$\lim_{j\to+\infty} \frac{1}{R_j^2} \sum_{B_{R_j}(q)} v^p(x) \leq \lim_{j\to+\infty} \frac{|B_{R_j}(q)|}{R_j^2} = B < +\infty.$$

Theorem A implies that v, and hence u, is constant. ∎

3. Non-existence of ℓ^p-subharmonic functions. The case $1 < p < 2$

The aim of this section is to follow the path previously undertaken and prove Theorem C concerning the case $1 < p < 2$. We note that conclusion (2.1) of Theorem A is now replaced by the weaker (3.1). However, this latter is enough to guarantee the validity of the natural generalization of Corollary B stated in Corollary D below.

We also point out that the technical device described in Lemma 3.2, which allows us to restrict our attention to the class of positive subharmonic functions, will also be used in the subsequent section 4.

THEOREM C: *Let u: $X \to [0, +\infty)$ be a non-constant, subharmonic function. Then, for any fixed vertex q and any $1 < p < 2$,*

$$\liminf_{R\to+\infty} \frac{1}{R^2} \sum_{B_R(q)} u^p(x) > 0. \tag{3.1}$$

In the proof of Theorem C we shall make use of

LEMMA 3.1: *Let u: $X \to [0, +\infty)$ be subharmonic and suppose there exist $p \in (1, 2)$ and $x \in X$ such that*

$$\Delta u^{p/2}(x) < 0. \tag{3.2}$$

Then there exists a constant $c_p \in (-\frac{1}{2}, 0)$ (depending only on p) such that

$$m(x) u^{p/2}(x) \Delta u^{p/2}(x) \geq c_p \left\| du^{p/2} \right\|^2 (x). \tag{3.3}$$

Proof: Obviously we have, from (3.2), $u(x) > 0$; (3.3) is equivalent to show that

$$u^{p/2}(x) \sum_{y\sim x} [u^{p/2}(x) - u^{p/2}(y)] \leq |c_p| \sum_{y\sim x} [u^{p/2}(x) - u^{p/2}(y)]^2.$$

Proceeding as in Lemma 2.2, since G has uniformly bounded vertex degree, it is enough to prove that, for $p \in (1,2)$ fixed, the supremum of the functions

$$\frac{\sum_{j=1}^{m}(1-z_j)}{\sum_{j=1}^{m}(1-z_j)^2}, \qquad m = m(x)$$

on

$$\left\{\sum_{j=1}^{m} z_j < m, \sum_{j=1}^{m} z_j^{2/p} \geq m, z_j \geq 0, j = 1, \dots, m\right\}$$

is strictly smaller than $\frac{1}{2}$. Towards this aim we use the inequality $z^\alpha \leq \alpha z + 1 - \alpha$, valid for $0 < \alpha < 1$, $z \geq 0$. We multiply by z, and get

$$z^2 \geq \frac{1}{\alpha} z^{1+\alpha} - \frac{1-\alpha}{\alpha} z.$$

Next, choose $1 + \alpha = 2/p$ and substitute for z_j^2 in the above. ∎

Proof (of Theorem C): We fix $q \in X$, $p \in (1,2)$ and take R sufficiently large so that

$$\left\|du^{p/2}\right\| \not\equiv 0 \quad \text{on } B_R(q) \tag{3.4}$$

(this is possible since u is non-constant). Let φ be a cut-off function for the region $B_R(q) = C_{R,0}(q)$. The divergence theorem and (1.14) give

$$\sum_{B_R(q)} m(x)\varphi^2(x)\Delta u^p(x) = -\frac{1}{2} \sum_{B_R(q)} (d\varphi^2 \cdot du^p)(x). \tag{3.5}$$

On the other hand,

$$\Delta u^p = \operatorname{div}(du^p) = \operatorname{div}(2u^{p/2} \star du^{p/2}) = 2u^{p/2}\Delta u^{p/2} + \frac{1}{m}\left\|du^{p/2}\right\|^2$$

and therefore (3.5) gives

$$\sum_{B_R(q)} \varphi^2(x)\left\|du^{p/2}\right\|^2(x) + 2\sum_{B_R(q)} m(x)\varphi^2(x)u^{p/2}(x)\Delta u^{p/2}(x) \leq \frac{1}{2}\sum_{B_R(q)} \left|(d\varphi^2 \cdot du^p)(x)\right|. \tag{3.6}$$

Next, we observe that, with a reasoning similar to that used in (2.23) of Step 1 in Theorem A, we have

$$\sum_{B_R(q)} \left|(d\varphi^2 \cdot du^p)(x)\right| \leq \sum_{B_R(q)} \left\|2\varphi \star du^{p/2}\right\|(x)\left\|2u^{p/2} \star d\varphi\right\|(x). \tag{3.7}$$

Hence, having arbitrarily fixed $\tau > 0$, (3.6) and (3.7) yield

$$\begin{aligned}(3.8)\qquad &\sum_{B_R(q)} \varphi^2(x) \left\|du^{p/2}\right\|^2(x) + 2\sum_{B_R(q)} m(x)\varphi^2(x)u^{p/2}(x)\Delta u^{p/2}(x)\\ &\leq \sum_{B_R(q)} \left(\tau \left\|u^{p/2} \star d\varphi\right\|^2(x) + \frac{1}{\tau}\left\|\varphi \star du^{p/2}\right\|^2(x)\right).\end{aligned}$$

We now use Lemma 2.1 (i), (ii) to estimate the RHS of (3.8) from above; thus

$$\begin{aligned}(3.9)\ &\left(1-\frac{1}{\tau}\right)\sum_{B_R(q)} \varphi^2(x) \left\|du^{p/2}\right\|^2(x) + 2\sum_{A_R(q)} m(x)\varphi^2(x)u^{p/2}(x)\Delta u^{p/2}(x)\\ &\leq \tau \sum_{B_R(q)} u^p(x) \left\|d\varphi\right\|^2(x)\end{aligned}$$

where $A_R(q) = \{x \in B_R(q)\colon \Delta u^{p/2}(x) < 0\}$.

Using the properties of the cut-off function φ and Lemma 3.1, from (3.9) we obtain

$$(3.10)\qquad \frac{M\tau}{R^2}\sum_{B_R(q)} u^p(x) \geq \left(1-\frac{1}{\tau}-2\left|c_p\right|\right)\sum_{B_R(q)} \varphi^2(x)\left\|du^{p/2}\right\|^2(x)$$

with $|c_p| < 1/2$. We may thus choose $\tau > 0$ such that

$$(3.11)\qquad 1-\frac{1}{\tau}-2\left|c_p\right| > 0.$$

Now (3.4), together with (3.10) and (3.11), imply the validity of (3.1). ∎

Remark: In (3.11) above we have used in an essential way $|c_p| < 1/2$. According to Lemma 3.1, this follows from the assumption $p > 1$. Indeed, for $p = 1$, (3.4) holds with $c_1 = -1/2$. Nevertheless, it is reasonable to wonder about the validity of Theorem C in this latter case. Unfortunately we have not been able to provide either a proof or a counterexample. However, we recall that, in the continuous case, there exist complete manifolds with non-constant subharmonic functions of class L^1 (see [L–S]).

COROLLARY D: *Let $0 < \delta < 1$ and assume that, for some (hence any) vertex q,*

$$(3.12)\qquad \liminf_{R\to+\infty} \frac{|B_R(q)|}{R^{2-\delta}} = 0.$$

Let $u\colon X \to \mathbf{R}$ be subharmonic and suppose that for some constant $A, B > 0$, $0 \leq k < \delta$,

$$(3.13)\qquad u(x) \leq A\rho^k(x,q) + B.$$

Then u is constant.

Remark: We wish to stress that nothing is required on the behaviour of u from below.

The following device allows us to reduce the proof of Corollary D to the case $u > 0$.

LEMMA 3.2: *Let u: $X \to \mathbb{R}$ be subharmonic. Then for each $a \in \mathbb{R}$ there exist positive constants A_a, B_a and a subharmonic function $v = v_a$: $X \to (0, +\infty)$ such that*

$$v(x) \leq A_a \text{ Max } \{u(x), a\} + B_a. \tag{3.14}$$

Furthermore, v is constant if and only if u is constant.

Proof: We fix $a \in \mathbb{R}$ and let

$$\chi_a(t) = \begin{cases} e^t & \text{if } t \leq a-1, \\ (a-t)e^{(a-1)} & \text{if } a-1 < t \leq a, \\ 0 & \text{if } t > a. \end{cases} \tag{3.15}$$

Next, we set

$$g(t) = \int_{-\infty}^{t} \int_{-\infty}^{s} \chi_a(\tau) d\tau ds. \tag{3.16}$$

Obviously g satisfies

$$\begin{cases} \text{(i) } g(t) > 0, \quad \text{(ii) } g'(t) > 0 \text{ on } \mathbb{R}; \\ \text{(iii) } g''(t) > 0 \text{ on } (-\infty, a), \quad g''(t) = \chi_a(t) = 0 \text{ on } [a, +\infty). \end{cases} \tag{3.17}$$

We define on X

$$v(x) = v_a(x) = g(u(x)). \tag{3.18}$$

Thus, from (3.17)(i) v is positive. To show that v is subharmonic we note that by Taylor's formula, for each $x \in X$ and $y \sim x$, there exists a value $\eta_{x,y}$ between $u(x)$ and $u(y)$ such that

$$\begin{aligned} m(x)\Delta v(x) &= \sum_{y\sim x} [v(y) - v(x)] \\ &= \sum_{y\sim x} \left\{ g'(u(x))[u(y) - u(x)] + \frac{1}{2} g''(\eta_{x,y})[u(y) - u(x)]^2 \right\} \\ &= m(x)g'(u(x))\Delta u(x) + \frac{1}{2} \sum_{y\sim x} g''(\eta_{x,y})[u(y) - u(x)]^2 \end{aligned}$$

and the desired property is then a consequence of (3.17) (ii) (iii) and $\Delta u \geq 0$. Finally (3.16) implies

$$g(t) \leq At + B \tag{3.19}$$

for some $A = A_a$, $B = B_a$ so that (3.14) follows at once. The last assertion of the lemma is due to (3.17) (ii). ∎

Proof (of Corollary D): We fix $a \in \mathbf{R}$ and let $v(x) = v_a(x)$ be the positive, subharmonic function defined in Lemma 3.2. Putting together (3.13) and (3.14) we have

$$v(x) \leq A'\rho^k(x,q) + B' \tag{3.20}$$

for some constants $A', B' > 0$. Next, we observe that for any $p \geq 1$,

$$\frac{1}{R^2}\sum_{B_R(q)} v^p(x) \leq c_1 \frac{|B_R(q)|}{R^2} + c_2 \frac{1}{R^2}\sum_{B_R(q)} \rho^{kp}(x,q)$$
$$\leq \frac{c}{R^2}\sum_{B_R(q)} \rho^{kp}(x,q) = \frac{c}{R^2}\sum_{j=1}^{R} j^{kp}\,|S_j(q)|.$$

Choosing $p = \delta/k > 1$, we obtain

$$\frac{1}{R^2}\sum_{B_R(q)} v^p(x) \leq \frac{c}{R^2}\sum_{j=1}^{R} j^{\delta}\,|S_j(q)| \leq \frac{c}{R^{2-\delta}}\sum_{j=1}^{R} |S_j(q)| \leq c\frac{|B_R(q)|}{R^{2-\delta}}.$$

From (3.12) it therefore follows that

$$\liminf_{R\to+\infty} \frac{1}{R^2}\sum_{B_R(q)} v^p(x) = 0.$$

Now, Theorem C, in case $p < 2$, or Theorem A otherwise, imply that v, and hence u, is constant. ∎

4. Subharmonic functions of logarithmic growth

In Corollaries B and D we have proven constancy of subharmonic functions which are either bounded above or of sufficientely small polynomial growth, provided, respectively, that

$$\text{(i)} \liminf_{R\to+\infty} \frac{|B_R(q)|}{R^2} < +\infty; \quad \text{or} \quad \text{(ii)} \liminf_{R\to+\infty} \frac{|B_R(q)|}{R^{2-\delta}} = 0 \tag{4.1}$$

for some $\delta > 0$.

It is interesting to analyse the case when the growth of u ranges between the behaviours above. Theorem E provides a satisfactory answer when u is at most of logarithmic growth. The proof we shall present below relies, in an essential way, on Lemma 4.1, which may be interesting on its own. We mention that this latter has been inspired by some work of Li and Tam, [L–T], in the continuous case.

THEOREM E: *Assume that, for some vertex $q \in X$,*

$$\lim_{R\to+\infty} \frac{|B_R(q)|}{R^2} = 0. \tag{4.2}$$

Let $u: X \to \mathbb{R}$ be subharmonic and suppose that

$$u(x) \leq A \log \rho(x,q) + B \tag{4.3}$$

for some positive constants A, B. Then u is constant.

The proof of Theorem E is based on the next result. First of all we introduce some notations. Given $v: X \to \mathbb{R}$, $q \in X$ and $k \in \mathbb{N}$, we set

$$s_k(v) = \operatorname*{Max}_{S_k(q)} v(x), \qquad i_k(v) = \operatorname*{Min}_{S_k(q)} v(x). \tag{4.4}$$

We are now ready to state

LEMMA 4.1: *Let $v: X \to [0, +\infty)$ be such that $\Delta v \geq 0$ and fix $q \in X$. Then, there exists a positive constant $c = c(G)$, depending only on the graph G, such that for each $R \geq r+3 \geq 4$:*

$$\sum_{x\in S_{r+1}(q)} \sum_{\substack{y\in S_r(q)\\ y\sim x}} [v(x) - v(y)] \leq cr \frac{[\beta_R - i_r(v)]^2}{\beta_R - M i_r(v)} \left(\frac{1}{\sum_{r+1}^R \frac{j}{|B_j(q)|}} \right) \tag{4.5}$$

provided $\beta_R > M i_r(v)$ with M as in (1.2), and $\beta_R \geq s_R(v)$.

Let us assume, for the moment, the validity of Lemma 4.1 and show how to deduce Theorem E from it.

Proof (of Theorem E): Fix a vertex $x_0 \in X$ and choose

$$a > \operatorname*{Max}_{y\sim x_0} \{u(x_0), u(y)\}. \tag{4.6}$$

Let $g(t) = g_a(t)$ be as in (3.16) of Lemma 3.2, and define

$$v(x) = g(u(x)). \tag{4.7}$$

Then v is positive, subharmonic and, because of (4.3),

$$v(x) \leq A_1 \log \rho(x,q) + B_1 \tag{4.8}$$

for some positive constants A_1, B_1. Now we fix $r > 1$. We can assume to have chosen R_0 sufficiently large so that

$$\text{(i)} A_1 \log R_0 + B_1 > Mi_r(v), \qquad \text{(ii)} R_0 > r+3. \tag{4.9}$$

We now put $R \geq R_0$. By Green's formula, subharmonicity of v, (4.8), (4.9) (i), (ii) and Lemma 4.1 we obtain

$$\begin{aligned} 0 \leq \sum_{B_r(q)} m(y)\Delta v(y) &= \sum_{y \in S_r(q)} \sum_{\substack{x \in S_{r+1}(q) \\ x \sim y}} [v(x) - v(y)] \\ &= \sum_{x \in S_{r+1}(q)} \sum_{\substack{y \in S_r(q) \\ y \sim x}} [v(x) - v(y)] \\ &\leq cr \frac{[A_1 \log R + B_1]^2}{A_1 \log R + B_1 - Mi_r(v)} \frac{1}{\sum_{r+1}^R \frac{j}{|B_j(q)|}} \leq c_1 \frac{\log R}{\sum_{r+1}^R \frac{j}{|B_j(q)|}} \end{aligned} \tag{4.10}$$

where the positive constant c_1 is independent of R, provided we have chosen R_0 sufficiently large.

We now claim that

$$\lim_{R \to +\infty} \frac{\log R}{\sum_{r+1}^R \frac{j}{|B_j(q)|}} = 0. \tag{4.11}$$

Indeed, we set $b_j = j / |B_j(q)|$ and observe that (4.2) imply

$$jb_j \to +\infty \qquad \text{as } j \to +\infty. \tag{4.12}$$

Thus, having fixed any positive constant A there exists $j_0 = j_0(A)$ such that

$$jb_j \geq A \qquad \text{for each } j \geq j_0. \tag{4.13}$$

Let $R > j_0$. We compute

$$\sum_{j=r+1}^{R} b_j \geq \sum_{j=r+1}^{j_0} b_j + A\sum_{j_0+1}^{R}\frac{1}{j} \geq \frac{A}{2}\log R - A\sum_{j=1}^{j_0}\frac{1}{j}.$$

Therefore

$$\frac{\sum_{j=r+1}^{R} b_j}{\log R} \geq \frac{A}{2} - A\frac{\sum_{j=1}^{j_0}\frac{1}{j}}{\log R}, \quad R > j_0 \tag{4.14}$$

showing the validity of (4.11). Putting together (4.10) and (4.11) we conclude that v is harmonic on $B_R(q)$ and hence on X, since the choice of r was arbitrary.

Having proved harmonicity of v, Taylor's formula gives

$$\begin{aligned} 0 &= m(x_0)\Delta v(x_0) \\ &= m(x_0)g'(u(x_0))\Delta u(x_0) + \frac{1}{2}\sum_{y\sim x_0} g''(\eta_{x_0,y})[u(x_0) - u(y)]^2 \end{aligned} \tag{4.15}$$

for appropriate values $\eta_{x_0,y}$ between $u(x_0)$ and $u(y)$. On the other hand, choice (4.6) of a implies $\eta_{x_0,y} < a$ for each $y \sim x_0$. Therefore, according to (3.17) (iii),

$$g''(\eta_{x_0,y}) > 0 \qquad \text{for each } y \sim x_0. \tag{4.16}$$

Since $g'(u(x_0)) > 0$ and $\Delta u(x_0) \geq 0$, (4.15) and (4.16) force $u(x_0) = u(y)$ for each $y \sim x_0$. Hence u is locally constant, and therefore constant because the graph is connected. ∎

Proof (of Lemma 4.1): To simplify notations we set

$$D = C_{R,r}(q) \qquad \text{and} \qquad \mathring{D} = C_{R-1,r+1}(q).$$

Let $f\colon D \to \mathbb{R}$ be the solution of the problem

$$\begin{cases} \Delta f(x) = 0 & \text{on } \mathring{D}, \\ f(x) = v(x) & \text{on } S_r(q), \\ f(x) = \beta_R & \text{on } S_R(q). \end{cases} \tag{4.17}$$

By the maximum principle $v \leq f$ on D, and hence

$$v(y) - v(x) \geq f(y) - f(x) \qquad \text{for } x \in S_{r+1}(q), \quad y \in S_r(q). \tag{4.18}$$

Similarly, let $h\colon D \to \mathbb{R}$ be the solution of

$$\begin{cases} \Delta h(x) = 0 & \text{on } \mathring{D}, \\ h(x) = i_r(v) & \text{on } S_r(q), \\ h(x) = \beta_R & \text{on } S_R(q). \end{cases} \tag{4.19}$$

Again, the maximum principle implies

$$h(y) - h(x) \geq f(y) - f(x) \qquad \text{for } x \in S_{R-1}(q), \quad y \in S_R(q). \tag{4.20}$$

Define

$$\Lambda_R = \{(x,y)\colon x \in S_{R-1}(q), y \in S_R(q), y \sim x\}$$

and

$$\Omega_r = \{(x,y)\colon x \in S_{r+1}(q), y \in S_r(q), y \sim x\}.$$

An application of Green's formula gives

$$0 = \sum_{\Lambda_R} df(x,y) + \sum_{\Omega_r} df(x,y) \quad \text{and} \quad 0 = \sum_{\Lambda_R} dh(x,y) + \sum_{\Omega_r} dh(x,y), \tag{4.21}$$

so that (4.21), (4.20) and (4.19) imply

$$\begin{aligned} -\sum_{\Omega_r} dv(x,y) &\leq -\sum_{\Omega_r} df(x,y) \\ &= \sum_{\Lambda_R} df(x,y) \leq \sum_{\Lambda_R} dh(x,y) = -\sum_{\Omega_r} dh(x,y). \end{aligned} \tag{4.22}$$

Next, we compare $-\sum_{\Omega_r} dh(x,y)$ with $\sum_{\mathring{D}} \|dh\|^2(x)$. Towards this aim we use Green's formula and (4.19) to compute

$$\begin{aligned} \sum_{\mathring{D}} \|dh\|^2(x) &= \sum_{x\in\mathring{D}} \sum_{y\sim x} [h(x) - h(y)]^2 \\ &= \sum_{\substack{x,y\in\mathring{D} \\ x\sim y}} [h(x) - h(y)]^2 + \sum_{x\in\mathring{D}} \sum_{\substack{y\in S_r(q)\cup S_R(q) \\ y\sim x}} [h(x) - h(y)]^2 \\ &= 2\sum_{x\in\mathring{D}} \sum_{\substack{y\in S_r(q)\cup S_R(q) \\ y\sim x}} h(x)[h(y) - h(x)] \\ &\quad + \sum_{x\in\mathring{D}} \sum_{\substack{y\in S_r(q)\cup S_R(q) \\ y\sim x}} [h(x) - h(y)]^2 \\ &= \sum_{x\in\mathring{D}} \sum_{\substack{y\in S_r(q)\cup S_R(q) \\ y\sim x}} [h(x) + h(y)][h(y) - h(x)]. \end{aligned} \tag{4.23}$$

Since h is constant on $S_R(q)$ we have

$$dh(x,y) \geq 0 \qquad \text{if } x \in S_{R-1}(q), \quad y \in S_R(q), \quad \text{and} \quad y \sim x; \tag{4.24}$$

similarly,

$$dh(x,y) \leq 0 \qquad \text{if } x \in S_{r+1}(q), \quad y \in S_r(q), \quad \text{and} \quad y \sim x. \tag{4.25}$$

Putting together (4.19) and (4.22)–(4.25) we obtain

$$\begin{aligned} \sum_{\overset{\circ}{D}} \|dh\|^2(x) &= \sum_{x\in S_{R-1}(q)} \sum_{\substack{y\in S_R(q)\\ y\sim x}} [h(x)+\beta_R]dh(x,y) \\ &\quad + \sum_{x\in S_{r+1}(q)} \sum_{\substack{y\in S_r(q)\\ y\sim x}} [h(x)+i_r(v)]dh(x,y) \\ &\geq [\beta_R + i_{R-1}(h)]\sum_{\Lambda_R} dh(x,y) + [i_r(v)+s_{r+1}(h)]\sum_{\Omega_r} dh(x,y) \\ &\geq [\beta_R + i_{R-1}(h) - i_r(v) - s_{r+1}(h)]\sum_{\Omega_r}[-dh(x,y)]. \end{aligned} \tag{4.26}$$

The maximum principle and $\beta_R \geq M i_r(v)$ give

$$\beta_R \geq s_{r+1}(h)$$

and from (4.26) we finally obtain

$$\sum_{\overset{\circ}{D}} \|dh\|^2(x) \geq [i_{R-1}(h) - i_r(v)]\sum_{\Omega_r}[-dh(x,y)]. \tag{4.27}$$

On the other hand, since h is non-negative and harmonic in $\overset{\circ}{D}$,

$$h(x) = \frac{1}{m(x)}\sum_{y\sim x} h(y) \geq \frac{\beta_R}{M} \qquad \text{for each } x \in S_{R-1}(q).$$

Hence, (4.27), (4.25) and (4.22) yield

$$-\sum_{\Omega_r} dv(x,y) \leq \frac{M}{\beta_R - M i_r(v)} \sum_{\overset{\circ}{D}} \|dh\|^2(x). \tag{4.28}$$

Next, we estimate the energy of h on $\overset{\circ}{D}$ from above with the aid of an auxiliary function. Thus we define the radial functions

$$\sigma(x) = \sigma(\rho(x,q)) = \sum_{j=r}^{\rho(x,q)} \frac{j}{|B_j(q)|} \qquad \text{on } D \tag{4.29}$$

and

$$\varphi(x) = a\sigma(x) + b \qquad \text{on } D \tag{4.30}$$

with

$$a = \frac{\beta_R - i_r(v)}{\sigma(R) - \sigma(r)} \quad \text{and} \quad b = \frac{i_r(v)\sigma(R) - \beta_R\sigma(r)}{\sigma(R) - \sigma(r)}. \tag{4.31}$$

The above choice of a and b implies

$$\varphi(x) = h(x) \qquad \text{on } S_r(q) \cup S_R(q).$$

Since harmonic functions minimize energy, (4.28) gives

$$-\sum_{\Omega_r} dv(x,y) \leq \frac{M}{\beta_R - Mi_r(v)} \sum_{\overset{\circ}{D}} \|d\varphi\|^2(x). \tag{4.32}$$

To compute the energy of φ let us set

$$\begin{aligned} \operatorname{out}(x) &= |\{y \sim x \colon \rho(y,q) = \rho(x,q) + 1\}| \quad \text{and} \\ \operatorname{in}(x) &= |\{y \sim x \colon \rho(y,q) = \rho(x,q) - 1\}|. \end{aligned}$$

Obviously

$$\operatorname{out}(x), \operatorname{in}(x) \leq M. \tag{4.33}$$

Furthermore, (1.2) implies $M^{-1}|S_{j+1}(q)| \leq |S_j(q)|$ so that

$$|B_{j+1}(q)| \leq (1+M)\,|B_j(q)|. \tag{4.34}$$

Hence, (4.33), (4.34), (4.30) and (4.29) yield

$$\begin{aligned}
\sum_{\overset{\circ}{D}} \|d\varphi\|^2(x) &= \sum_{j=r+1}^{R-1} \sum_{x\in S_j(q)} \|d\varphi\|^2(x) \\
&= \sum_{j=r+1}^{R-1} \sum_{x\in S_j(q)} \sum_{y\sim x}[\varphi(y)-\varphi(x)]^2 = a^2 \sum_{j=r+1}^{R-1} \sum_{x\in S_j(q)} \sum_{y\sim x}[\sigma(y)-\sigma(x)]^2 \\
&= a^2 \sum_{j=r+1}^{R-1} \sum_{x\in S_j(q)} \left\{ \operatorname{out}(x)\frac{(j+1)^2}{|B_{j+1}(q)|^2} + \operatorname{in}(x)\frac{j^2}{|B_j(q)|^2} \right\} \\
&\le a^2 M \sum_{j=r+1}^{R-1} |S_j(q)| \left\{ \frac{(j+1)^2}{|B_{j+1}(q)|^2} + \frac{j^2}{|B_j(q)|^2} \right\} \le 8a^2 M \sum_{j=r+1}^{R} \frac{j^2 |S_j(q)|}{|B_j(q)|^2} \\
&= 8a^2 M \sum_{j=r+1}^{R} j^2 \frac{(|B_j(q)| - |B_{j-1}(q)|)}{|B_j(q)|^2} \\
&\le 8a^2 M \sum_{j=r+1}^{R} j^2 \left\{ \frac{1}{|B_{j-1}(q)|} - \frac{1}{|B_j(q)|} \right\} \\
&\le 8a^2 M \left\{ \frac{(r+1)^2}{|B_r(q)|} + \sum_{j=r+1}^{R-1} \frac{2j+1}{|B_j(q)|} \right\} \\
&\le 8a^2 M \left\{ \frac{(r+1)^2(M+1)}{|B_{r+1}(q)|} + 3\sum_{j=r+1}^{R-1} \frac{j}{|B_j(q)|} \right\} \le cra^2 \sum_{j=r+1}^{R} \frac{j}{|B_j(q)|}
\end{aligned} \tag{4.35}$$

where the constant c depends only on the geometric structure of the graph G.

Putting together (4.32), (4.31) and (4.35) we deduce

$$\begin{aligned}
-\sum_{\Omega_r} dv(x,y) &\le cra^2 \sum_{r+1}^{R} \frac{j}{|B_j(q)|}[\beta_R - Mi_r(v)]^{-1} \\
&= cr\frac{[\beta_R - i_r(v)]^2}{\beta_R - Mi_r(v)} \frac{1}{\sum_{r+1}^{R} \frac{j}{|B_j(q)|}},
\end{aligned}$$

that is, (4.5). ∎

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